Math 1510 Week 2

Sequence of real numbers

A sequence $\{a_n\}$ consists of real numbers

$$a_1, a_2, a_3, a_4, \cdots$$

Equivalently, it is a function from $\mathbb{N}$ to $\mathbb{R}$

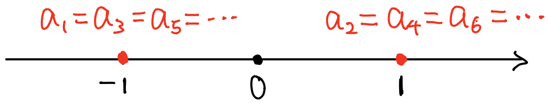
eg1 $a_n = (-1)^n$

$$a_1 = a_3 = a_5 = \cdots = -1$$

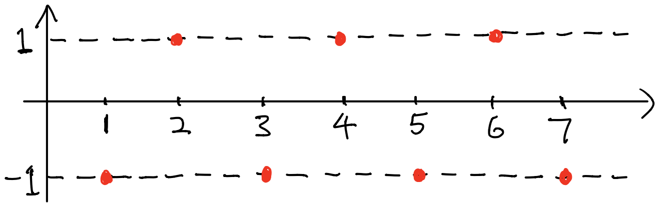
$$a_2 = a_4 = a_6 = \cdots = 1$$

$$-1, 1, -1, 1, -1, \cdots$$

Picture 1



Picture 2

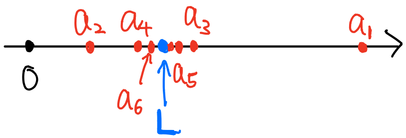


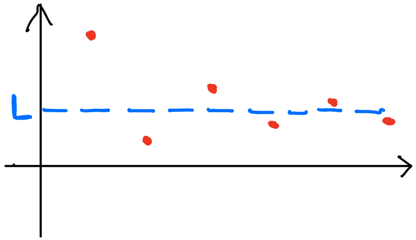
eg2 (Recursive Sequence)

$$a_1 = 2, \quad a_n = \frac{1}{1+a_{n-1}} \text{ for } n \geq 2$$

$$\Rightarrow a_2 = \frac{1}{1+2} = \frac{1}{3} \quad a_3 = \frac{1}{1+\frac{1}{3}} = \frac{3}{4} \quad a_4 = \frac{4}{7} \cdots$$

Pictures





Fact As n increases, a_n gets closer to L

where $L = \frac{\sqrt{5}-1}{2}$

Limit of Sequence

Intuitive Definition

Let $\{a_n\}$ be a sequence, $L \in \mathbb{R}$

- If a_n is close enough to L when n is large enough, then we say $\{a_n\}$ is a convergent sequence with

$$\lim_{n \to \infty} a_n = L$$

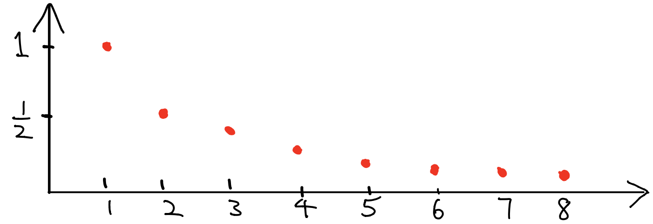
Another notation: $a_n \to L$ as $n \to \infty$

($\longrightarrow$ means approach)

- If no such L exists, then we say $\{a_n\}$ is a divergent sequence and

$\lim_{n \to \infty} a_n$ does not exist (DNE)

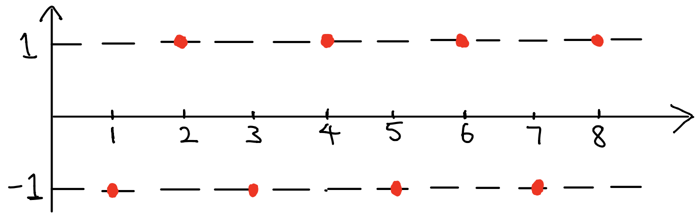
eg 1 Let $a_n = \frac{1}{n}$



As $n \to \infty$, $a_n \to 0$ (close to y-axis)

$\therefore \lim_{n \to \infty} a_n = 0$ (convergent)

eg 2 Let $a_n = (-1)^n$



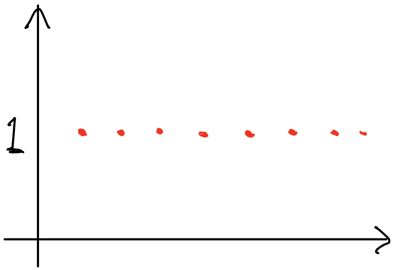
a_n doesn't approach a particular number as $n \to \infty$

$\therefore \lim_{n \to \infty} a_n$ DNE (divergent)

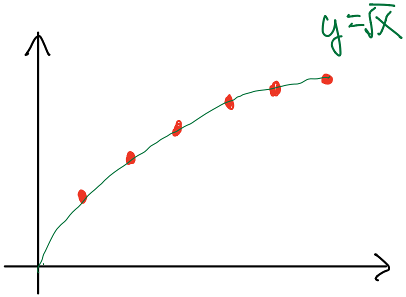
Rmk $\lim_{n \to \infty} a_{2n} = 1$, $\lim_{n \to \infty} a_{2n-1} = -1$

eg 3 Let $a_n = n^k$, where $k \in \mathbb{R}$ is a constant.

$$\lim_{n\to\infty} a_n = \begin{cases} +\infty \text{ (DNE)} & \text{if } k>0 \\ 1 & \text{if } k=0 \\ 0 & \text{if } k<0 \end{cases}$$

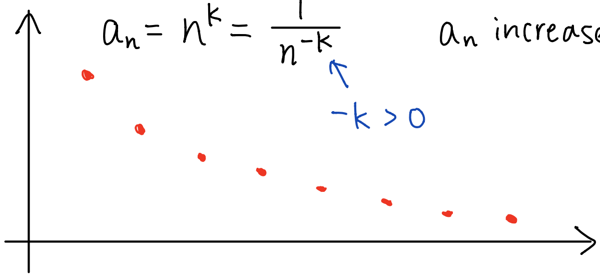


$k=0 \qquad a_n = 1$



$k=\frac{1}{2} \qquad a_n=\sqrt{n}$

As $n\to\infty$,
a_n increases to infinity



$a_n = n^k = \frac{1}{n^{-k}}$

$-k>0$

$k<0$

eg 4 $b\in\mathbb{R}$ is a constant. Let $a_n = b^n$.

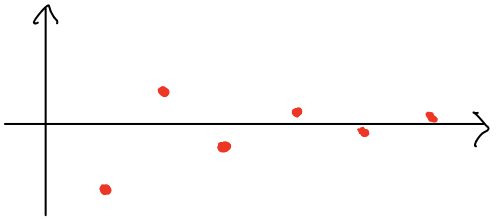
i.e. $b, b^2, b^3, b^4, \cdots$

$b=\frac{1}{2} \Rightarrow \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$

$b=-\frac{1}{2} \Rightarrow -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \cdots$

$b=-2 \Rightarrow -2, 4, -8, 16, \cdots$

$$\lim_{n\to\infty} a_n = \begin{cases} +\infty \text{ (DNE)} & \text{if } b>1 \\ 1 & \text{if } b=1 \\ 0 & \text{if } -1<b<1 \\ \text{DNE} & \text{if } b\le -1 \end{cases}$$



$b=\left(-\frac{1}{2}\right)^n$

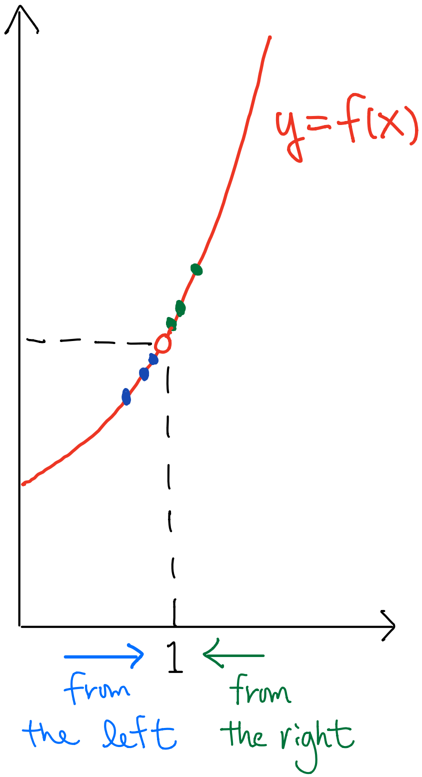
Formal definition of Limit (Not for EXAM)

$\lim_{n\to\infty} a_n = L \iff$ For any $\varepsilon>0$, there exists $N>0$ such that $|a_n - L|<\varepsilon$ for any $n>N$.

Limit of a function

Let $f(x) = \frac{x^3 - 1}{x - 1}$ $D_f = \mathbb{R} \setminus \{1\}$

Q Behavior of f(x) when x is near 1?



Observation

- For $x < 1$,

x	0.9	0.99	0.999
f(x)	2.71	2.9701	2.997001

As $x \to 1$ from the left, $f(x) \to 3$

We say that $\lim_{x \to 1^-} f(x) = 3$

- For $x > 1$

x	1.1	1.01	1.001
f(x)	3.31	3.0301	3.003001

As $x \to 1$ from the right, $f(x) \to 3$

We say that $\lim_{x \to 1^+} f(x) = 3$

- Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 3$,

we can collectively say that

$$\lim_{x \to 1} f(x) = 3$$

Intuitive definition for limit of functions

Let $a, L \in \mathbb{R}$, $f(x)$ be a function. We say

$$\left\{\begin{array}{l} \lim_{x\to a^-} f(x) = L \\ \lim_{x\to a^+} f(x) = L \\ \lim_{x\to a} f(x) = L \end{array}\right.$$ if $f(x)$ is close enough to L

when x is close enough to a with $\left\{\begin{array}{l} x < a \\ x > a \\ x \neq a \end{array}\right.$

Rmk

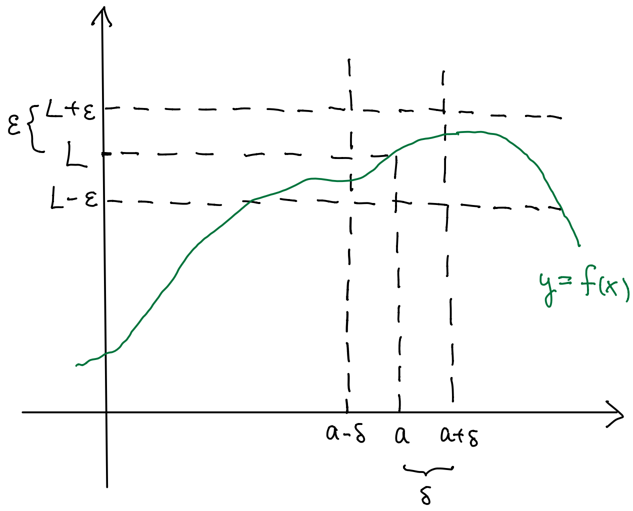
- The value $f(a)$ or whether $a \in D_f$ is not important for $\lim_{x\to a^\pm} f(x)$ or $\lim_{x\to a} f(x)$
- $$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Formal definition of limit of functions

(NOT for EXAM)

$\lim_{x\to a} f(x) = L$

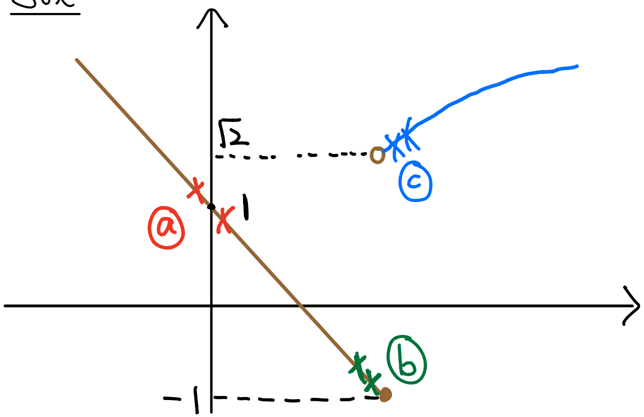
$\iff$ For any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \neq a$ and $|x-a| < \delta$ then $|f(x) - L| < \varepsilon$



eg Let $f(x) = \begin{cases} 1-x & \text{if } x \leq 2 \\ \sqrt{x} & \text{if } x > 2 \end{cases}$

Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 2} f(x)$.

Sol



ⓐ When x is near 0 and $x \neq 0$

$f(x) = 1 - x$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} 1 - x = 1$

ⓑ When x is near 2 and $x < 2$

$f(x) = 1 - x$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 1 - x = 1 - 2 = -1$

ⓒ When x is near 2 and $x > 2$

$f(x) = \sqrt{x}$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \sqrt{x} = \sqrt{2}$

$\because \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$

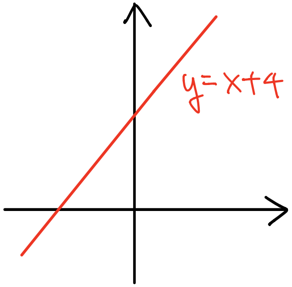
$\therefore \lim_{x \to 2} f(x)$ DNE

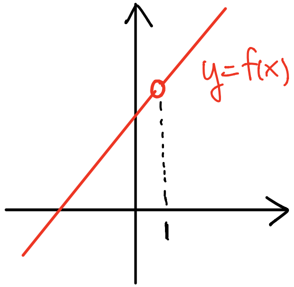
eg Let $f(x) = \dfrac{x^2+3x-4}{x-1}$ $\lim_{x \to 1} f(x) = ?$

Rmk $1 \notin D_f$ (Not important for $\lim_{x \to 1}$)

Sol Note if $x \neq 1$

$$f(x) = \frac{(x-1)(x+4)}{x-1} = x+4$$





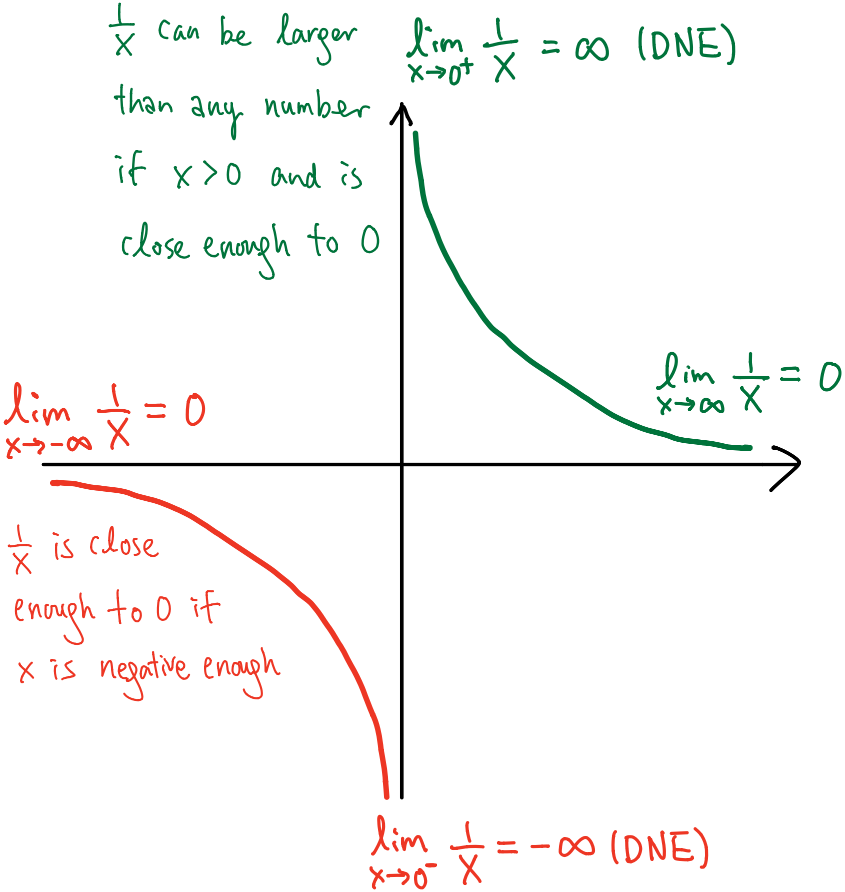
$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{(x-1)(x+4)}{x-1}$$

$$= \lim_{x \to 1} x+4$$

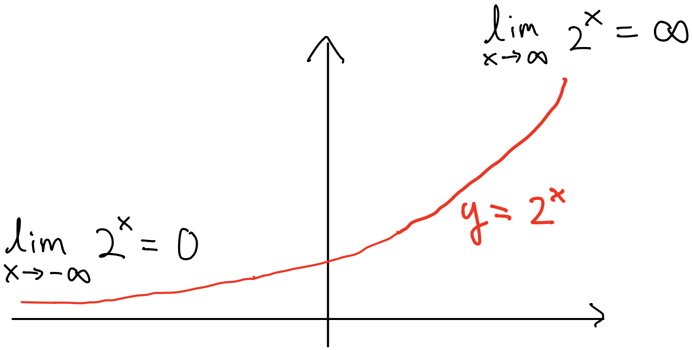
$$= 5$$

Limit involving infinities

eg. $f(x) = \dfrac{1}{x}$



eg $y = 2^x$



Similarly for any $a > 1$,

$$\lim_{x\to\infty} a^x = \infty \ (\text{DNE})$$

$$\lim_{x\to-\infty} a^x = 0$$

eg $\lim_{n\to\infty} 3^{n^2+n+1} = ?$

Sol As $n \to \infty$, $n^2+n+1 \to \infty$

$\because \lim_{y\to\infty} 3^y = \infty$

$\therefore \lim_{n\to\infty} 3^{n^2+n+1} = \infty \ (\text{DNE})$

eg $\lim_{n\to\infty} \left(\frac{1}{3}\right)^{n^2+n+1}$

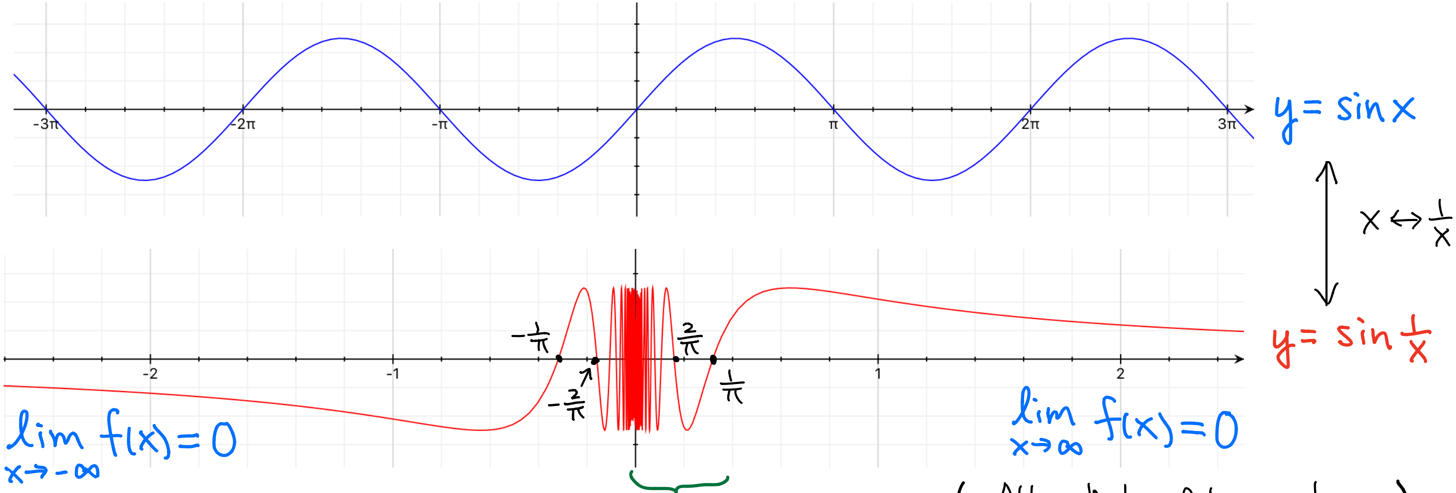
$$= \lim_{n\to\infty} \frac{1}{3^{n^2+n+1}}$$

$$= 0$$

eg $f(x) = \sin\frac{1}{x}$. Find $\lim_{x\to 0} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$.

Note $\sin x = 0$ at $x = 0, \pm\pi, \pm 2\pi, \cdots$

$\Rightarrow f(x) = \sin\frac{1}{x} = 0$ at $x = \pm\frac{1}{\pi}, \pm\frac{1}{2\pi}, \pm\frac{1}{3\pi}, \cdots$



As $x \to 0$, $f(x)$ keeps varying between -1 and 1

$\therefore \lim_{x\to 0} f(x)$ DNE

Similarly, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$ DNE

(Alternatively, let $y = \frac{1}{x}$

As $x \to \infty$, $y \to 0^+$

$\therefore \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \sin\frac{1}{x}$

$= \lim_{y\to 0^+} \sin y = 0$)

Basic Properties of Limits

Let $a \in \mathbb{R}$ or $\pm\infty$.

If both $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ exist, then

① $\lim_{x\to a} k = k$

② $\lim_{x\to a} [f(x) \pm g(x)] = \left(\lim_{x\to a} f(x)\right) \pm \left(\lim_{x\to a} g(x)\right)$

③ $\lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$

④ $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

⑤ $\lim_{x\to a} f(x)^k = \left(\lim_{x\to a} f(x)\right)^k$

⑥ $\lim_{x\to a} k^{f(x)} = k^{\lim_{x\to a} f(x)}$

} if defined

eg $(-1)^{\frac{1}{4}}$ is not defined

Rmk

i Similar results for one-sided limits and sequences

eg $\lim_{x\to a^+} f(x)g(x) = \left(\lim_{x\to a^+} f(x)\right)\left(\lim_{x\to a^+} g(x)\right)$

$\lim_{n\to\infty} (a_n - b_n) = \left(\lim_{n\to\infty} a_n\right) - \left(\lim_{n\to\infty} b_n\right)$

ii For limit $= \pm\infty$ and $L \in \mathbb{R}$,

the following ideas are intuitively true. However,

DO NOT write them down as they are informal

$\infty \pm L = \infty$

$-\infty - \infty = -\infty$

$\infty + \infty = \infty$

$-\infty - \infty = -\infty$

$L \cdot \infty = \begin{cases} +\infty & \text{if } L > 0 \\ -\infty & \text{if } L < 0 \end{cases}$

$\infty \cdot \infty = \infty$

$\frac{L}{\pm\infty} = 0$

iii Indeterminate forms

eg $\infty - \infty$, $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, 1^∞, ∞^0, 0^0

(limit $= 0$, not exactly zero)

eg. Find limits

① $\lim_{x\to\infty} \frac{2x^2+x-1}{3x^2+7}$ ← deg 2 (top), ← deg 2 (bottom) $\frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{2+\frac{1}{x}-\frac{1}{x^2}}{3+\frac{7}{x^2}}$

$= \frac{\lim_{x\to\infty}\left(2+\frac{1}{x}-\frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(3+\frac{7}{x^2}\right)}$

$= \frac{\lim_{x\to\infty} 2 + \lim_{x\to\infty}\frac{1}{x} - \lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 3 + \lim_{x\to\infty}\frac{7}{x^2}}$

$= \frac{2+0-0}{3+0}$

$= \frac{2}{3}$ (Top degree = Bottom degree)

② $\lim_{x\to\infty} \frac{1+x}{1-x^3}$ ← deg = 1 (top), ← (bottom)

$= \lim_{x\to\infty} \frac{\frac{1}{x^3}+\frac{1}{x^2}}{\frac{1}{x^3}-1}$

$= \frac{0+0}{0-1}$

$= 0$

(Bottom degree higher)

③ $\lim_{x\to-\infty} \frac{x^2+3}{x-7}$ $\left(\frac{\infty}{\infty}\right)$

~~$= \lim_{x\to-\infty} \frac{1+\frac{3}{x^2}}{\frac{1}{x}-\frac{7}{x^2}}$~~ Divide by x^2

~~$= \frac{1+0}{0-0}$~~ $\frac{1}{0}$ $+\infty, -\infty$? Not good

$\lim_{x\to-\infty} \frac{x^2+3}{x-7}$

Divide by x

$= \lim_{x\to-\infty} \frac{x+\frac{3}{x}}{1-\frac{7}{x}}$

∵ As $x\to-\infty$,
$x+\frac{3}{x} \to -\infty$
$1-\frac{7}{x} \to 1$

$= -\infty$ (DNE)

(Top degree higher)

For limits of rational function at $\pm\infty$, compare degrees of top and bottom.

④ $\lim_{x\to 0} \frac{1}{2x} - \frac{1}{x^2+2x}$ $(\pm\infty \pm \infty)$

$= \lim_{x\to 0} \frac{x^2+2x-2x}{(2x)(x^2+2x)}$

$= \lim_{x\to 0} \frac{x^2}{(2x)(x^2+2x)}$

$= \lim_{x\to 0} \frac{1}{2(x+2)}$

$= \frac{1}{2(0+2)}$

$= \frac{1}{4}$

⑤ $\lim_{x\to -2} \frac{x^3+8}{x^2-4}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to -2} \frac{(x+2)(x^2-2x+4)}{(x+2)(x-2)}$

$= \lim_{x\to -2} \frac{x^2-2x+4}{x-2}$

$= \frac{(-2)^2-2(-2)+4}{-2-2}$

$= \frac{12}{-4}$

$= -3$

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$
$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

⑥ $\lim_{x\to 1^+} \frac{1-x^3}{|1-x|}$ $\left(\frac{0}{0}\right)$

Note: For $x>1$,

$1-x<0$

$\Rightarrow |1-x| = -(1-x) = x-1$

$\therefore \lim_{x\to 1^+} \frac{1-x^3}{|1-x|}$

$= \lim_{x\to 1^+} \frac{1-x^3}{x-1}$

$= \lim_{x\to 1^+} \frac{(1-x)(1+x+x^2)}{x-1}$

$= \lim_{x\to 1^+} -(1+x+x^2)$

$= -3$

⑥ $$\lim_{x\to 2} \frac{2-x}{3-\sqrt{x^2+5}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 2} \frac{2-x}{3-\sqrt{x^2+5}} \cdot \frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}}$$

$$= \lim_{x\to 2} \frac{(2-x)(3+\sqrt{x^2+5})}{3^2-(x^2+5)}$$

$$= \lim_{x\to 2} \frac{(2-x)(3+\sqrt{x^2+5})}{4-x^2}$$

$$= \lim_{x\to 2} \frac{(2-x)(3+\sqrt{x^2+5})}{(2-x)(2+x)}$$

$$= \lim_{x\to 2} \frac{3+\sqrt{x^2+5}}{2+x}$$

$$= \frac{3+\sqrt{2^2+5}}{2+2} = \frac{3}{2}$$

⑦ $$\lim_{n\to\infty} \left(\sqrt{n^2+8n} - n\right) \qquad (\infty - \infty)$$

$$= \lim_{n\to\infty} \frac{\sqrt{n^2+8n}-n}{1} \cdot \frac{\sqrt{n^2+8n}+n}{\sqrt{n^2+8n}+n}$$

$$= \lim_{n\to\infty} \frac{n^2+8n-n^2}{\sqrt{n^2+8n}+n}$$

$$= \lim_{n\to\infty} \frac{8n}{\sqrt{n^2+8n}+n} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \frac{8}{\sqrt{1+\frac{8}{n}}+1}$$

$$= \frac{8}{\sqrt{1+0}+1}$$

$$= 4$$

⑧ $\lim_{x\to-\infty} 5x^3-100x^2-1000x+1$

$= \lim_{\to-\infty} x^3\left(5-\frac{100}{x}-\frac{1000}{x^2}+\frac{1}{x^3}\right)$

$= -\infty$ (DNE)

$\uparrow$

$\because \lim_{n\to-\infty} x^3 = -\infty$

$\lim_{n\to-\infty} 5-\frac{100}{x}-\frac{1000}{x^2}+\frac{1}{x^3} = 5$

⑨ $\lim_{x\to\infty} \ln\left(\frac{x}{1+x^2}\right)$

Let $y=\frac{x}{1+x^2}$,

As $x\to\infty$,

$y=\frac{x}{1+x^2}=\frac{\frac{1}{x}}{\frac{1}{x^2}+1}\to 0^+$

$\therefore \lim_{x\to\infty} \ln\left(\frac{x}{1+x^2}\right)$

$= \lim_{y\to0^+} \ln y$

$= -\infty$ (DNE)